CoE 163

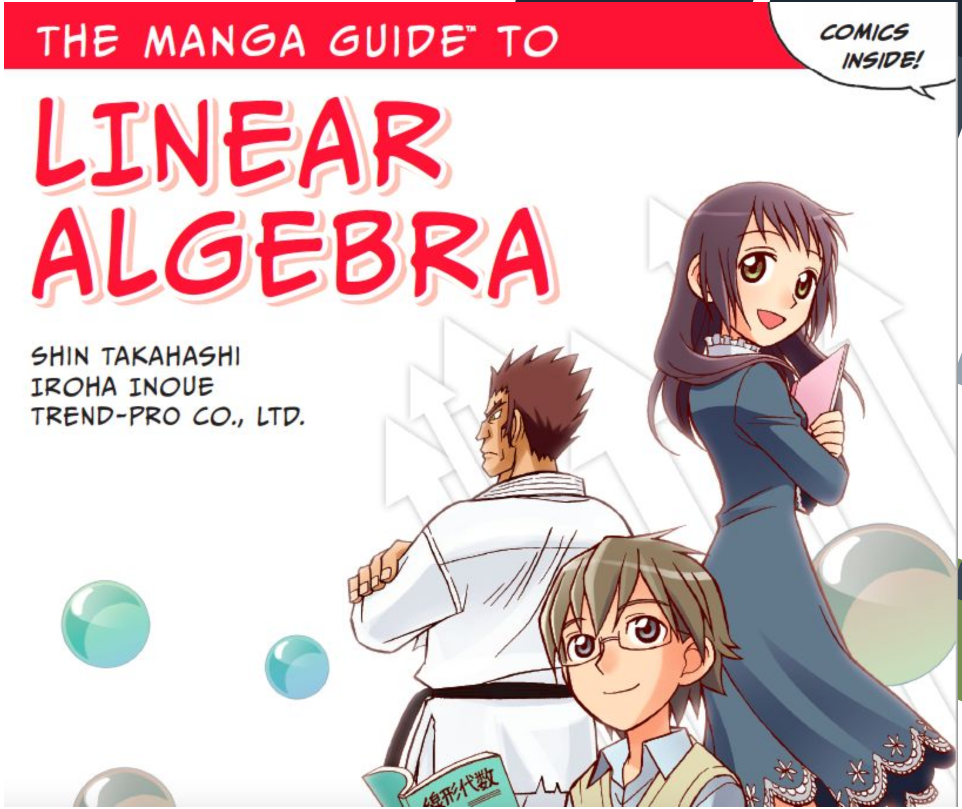
Computing Architectures and Algorithms

Review of Linear Algebra Operations

What is linear algebra?

Would like to acknowledge and thank this book:

Takahashi S, Inoue I, Lindh F, Co T. 2012. The Manga Guide to Linear Algebra. No Starch Press (Manga guide series).



Reiji Yurino was asked by his karate sensei to tutor the sensei's sister, Misa Ichinose

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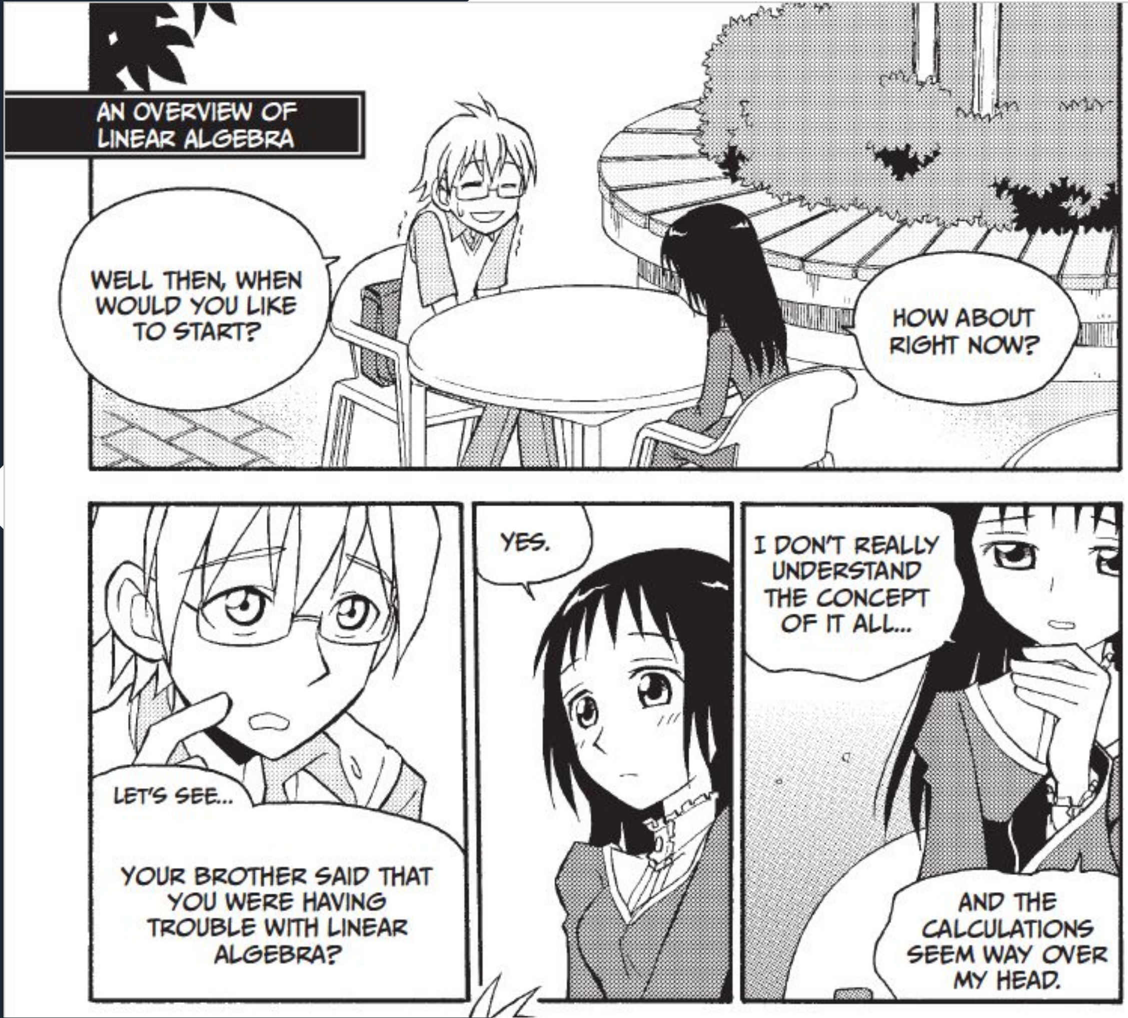

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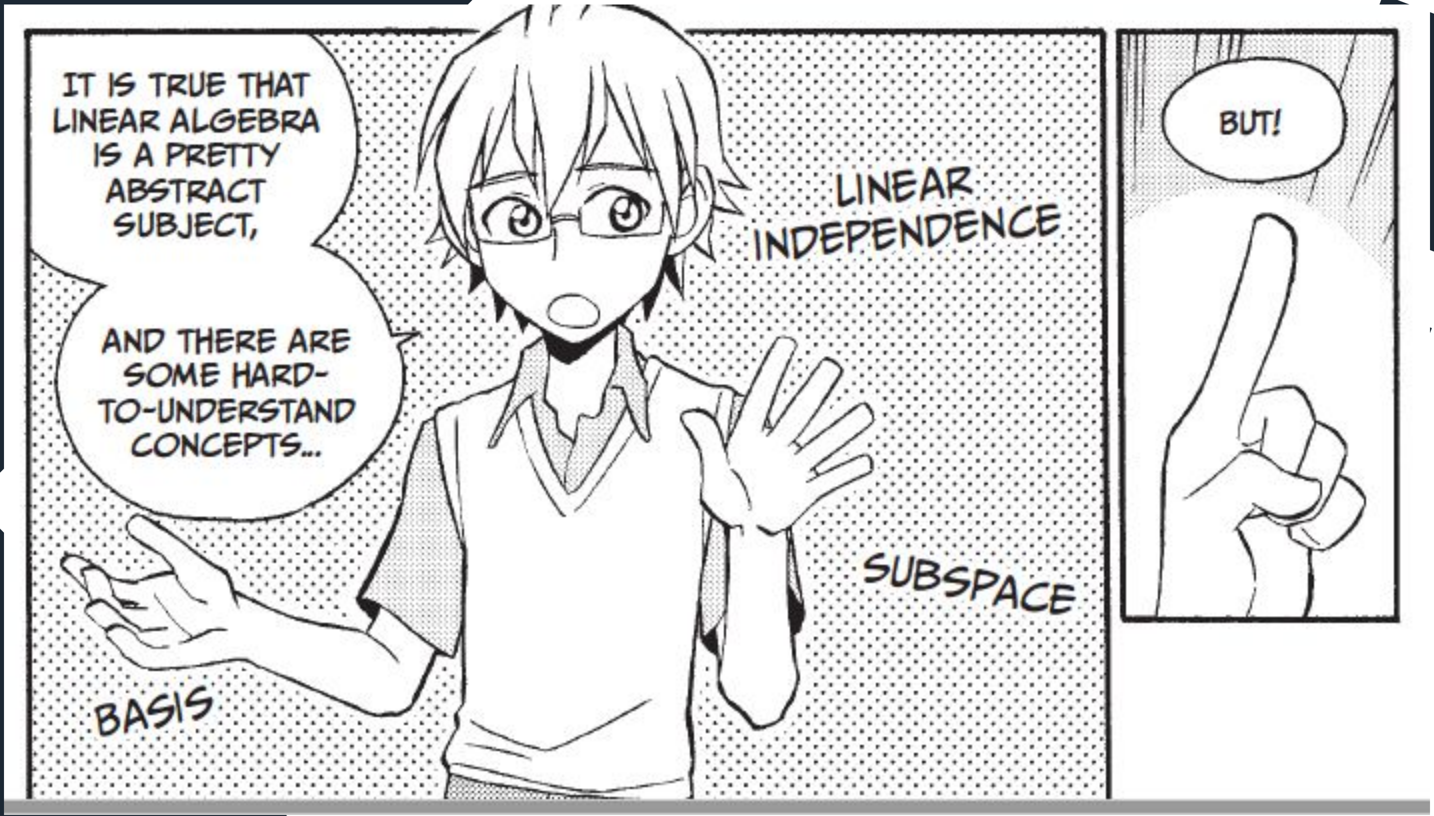

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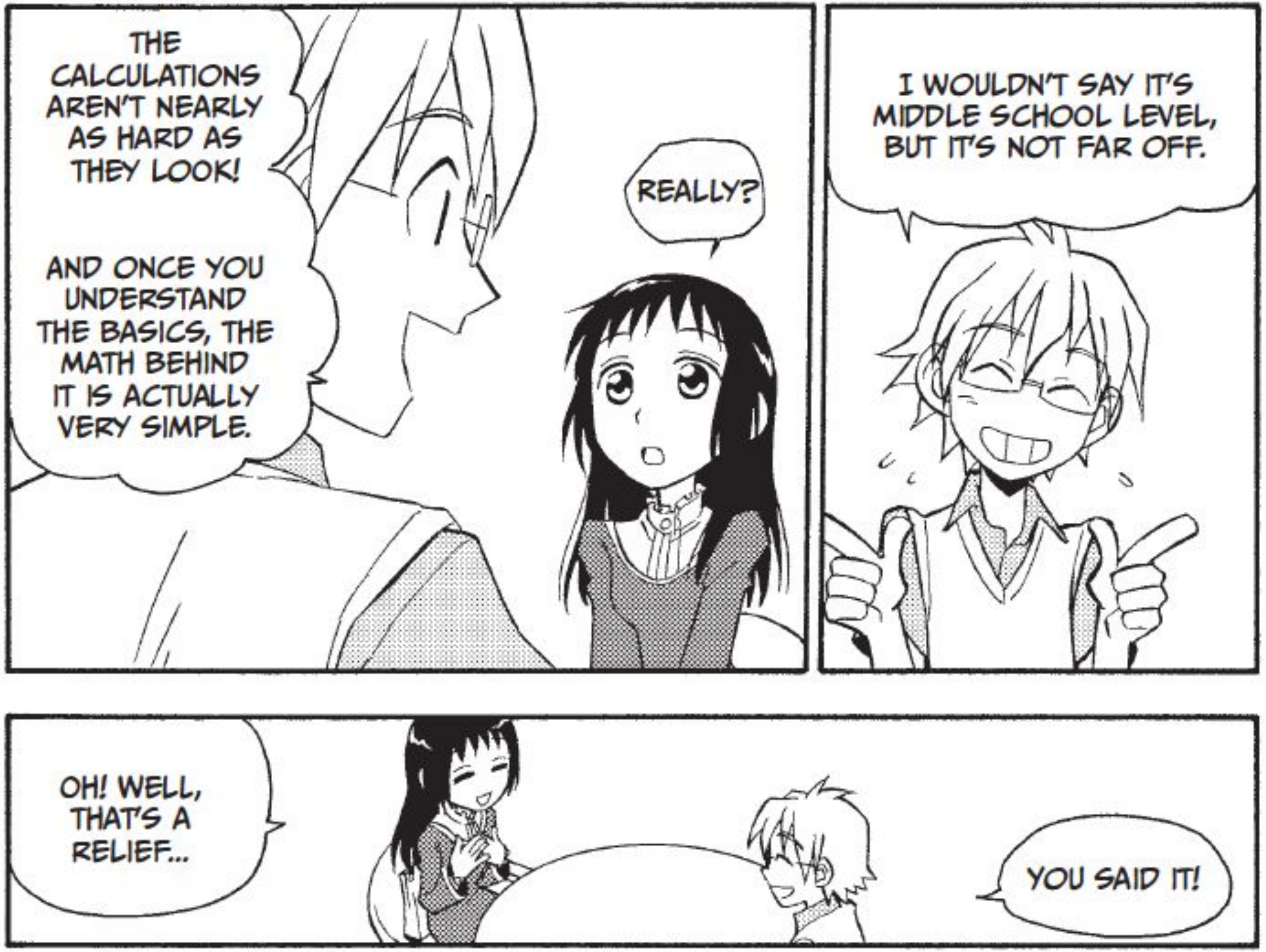

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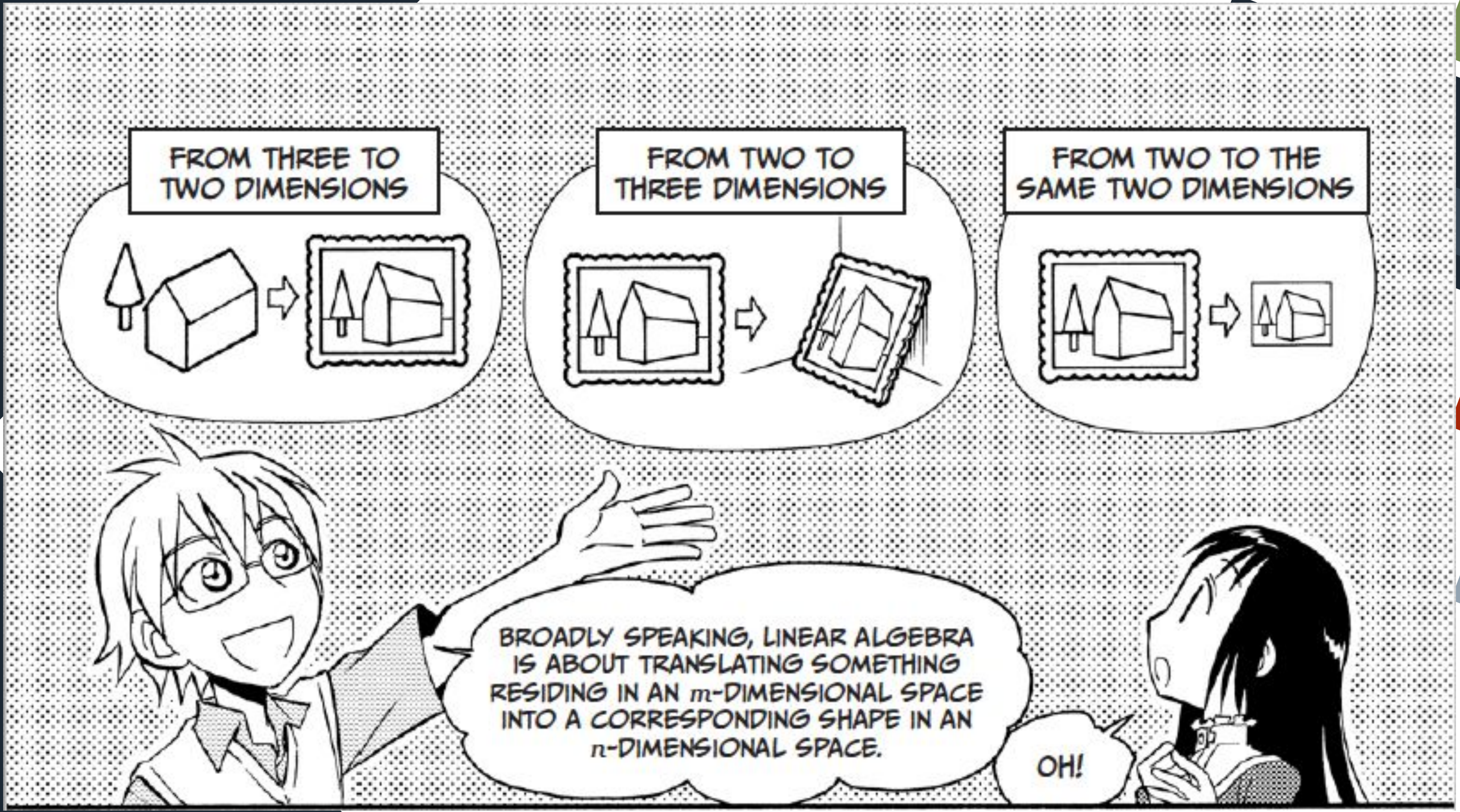

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Linear Algebra

- Branch of mathematics that deals with linear equations
  - Representation in vector spaces
  - Representation in matrices

Typical application: solving linear systems of equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1N}x_N &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2N}x_N &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3N}x_N &= b_3 \\
\cdots \qquad & \cdots \\
a_{M1}x_1 + a_{M2}x_2 + a_{M3}x_3 + \cdots + a_{MN}x_N &= b_M
\end{aligned}
$$

- N unknowns: x_j, $j = 1, 2, \ldots, N$
- M equations
- Known numbers:
  - Coefficients a_{ij}, $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$
  - b_i, $i = 1, 2, \ldots, M$

Typical application: solving linear systems of equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1N}x_N &= b_1 \\
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a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3N}x_N &= b_3 \\
\cdots \quad & \cdots \\
a_{M1}x_1 + a_{M2}x_2 + a_{M3}x_3 + \cdots + a_{MN}x_N &= b_M
\end{aligned}
$$

- Can be written in matrix form

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

Matrix representation of linear system of equations

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

Matrix of coefficients, $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & \cdots & & \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix}$$

Unknowns, $\mathbf{x}$
column vector with
N entries

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{bmatrix}$$

Right-hand side, $\boldsymbol{b}$
column vector with
M entries

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_M \end{bmatrix}$$

Operations that we will focus on in this course

- Matrix-matrix multiplication
- Gaussian elimination
- Matrix inversion
- Matrix decomposition

Matrix-Matrix Multiplication

THE PRODUCT $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} 1x_1 + 2x_2 & 1y_1 + 2y_2 \\ 3x_1 + 4x_2 & 3y_1 + 4y_2 \\ 5x_1 + 6x_2 & 5y_1 + 6y_2 \end{pmatrix}$

CAN BE DERIVED BY TEMPORARILY SEPARATING THE TWO COLUMNS $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ AND $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, FORMING THE TWO PRODUCTS

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{pmatrix} \quad \text{AND} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1y_1 + 2y_2 \\ 3y_1 + 4y_2 \\ 5y_1 + 6y_2 \end{pmatrix}$$

AND THEN REJOINING THE RESULTING COLUMNS:

$$\begin{pmatrix} 1x_1 + 2x_2 & 1y_1 + 2y_2 \\ 3x_1 + 4x_2 & 3y_1 + 4y_2 \\ 5x_1 + 6x_2 & 5y_1 + 6y_2 \end{pmatrix}$$

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- $$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 \cdot 3 + (-3) \cdot 1 & 8 \cdot 1 + (-3) \cdot 2 \\ 2 \cdot 3 + 1 \cdot 1 & 2 \cdot 1 + 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 24 - 3 & 8 - 6 \\ 6 + 1 & 2 + 2 \end{pmatrix} = \begin{pmatrix} 21 & 2 \\ 7 & 4 \end{pmatrix}$$

- $$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 8 + 1 \cdot 2 & 3 \cdot (-3) + 1 \cdot 1 \\ 1 \cdot 8 + 2 \cdot 2 & 1 \cdot (-3) + 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 24 + 2 & -9 + 1 \\ 8 + 4 & -3 + 2 \end{pmatrix} = \begin{pmatrix} 26 & -8 \\ 12 & -1 \end{pmatrix}$$

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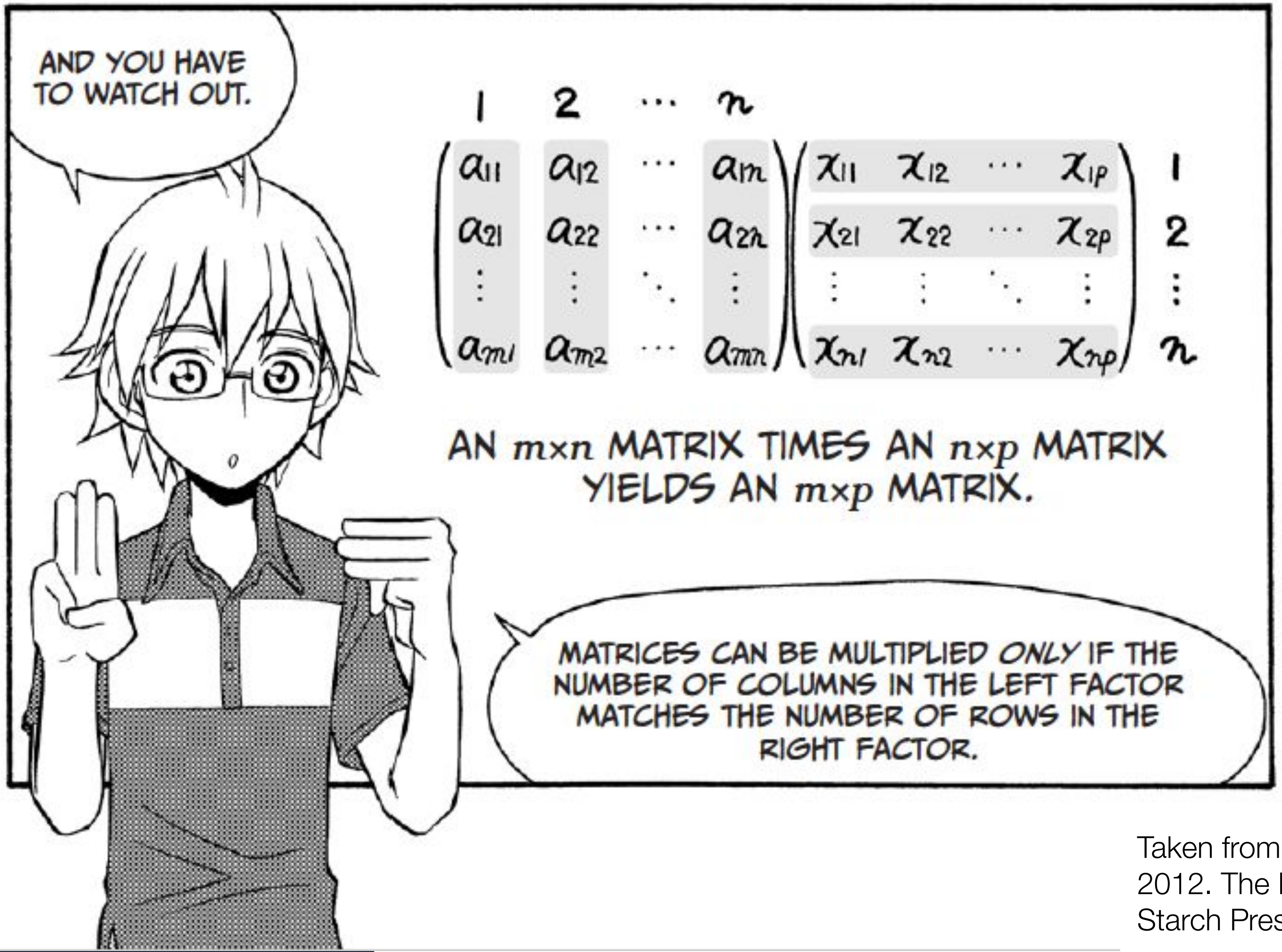

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Gaussian Elimination

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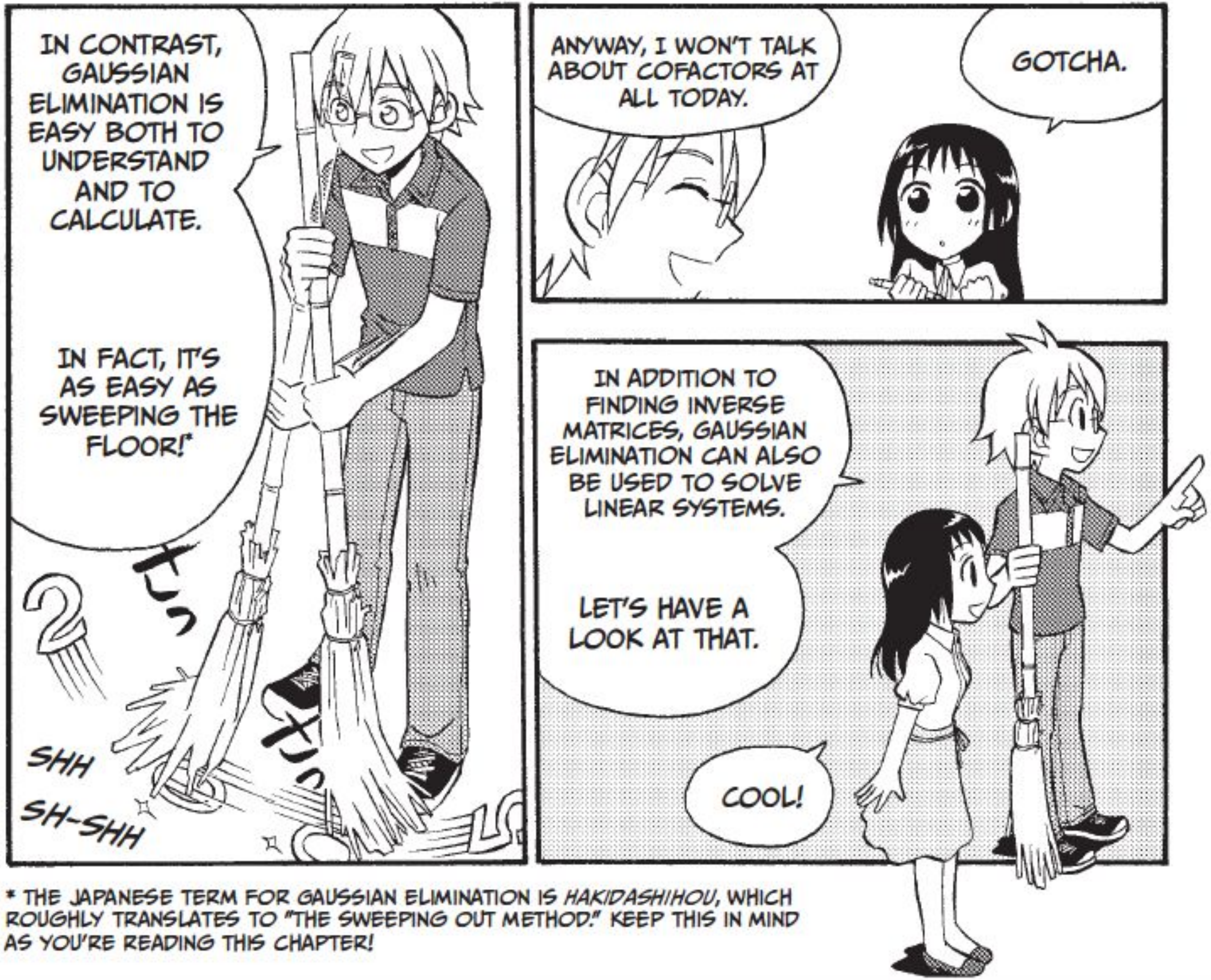

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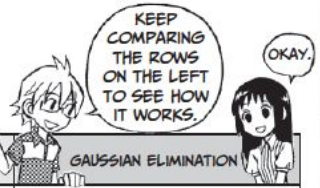
PROBLEM

Solve the following linear system:

$$\begin{cases} 3x_1 + 1x_2 = 1 \\ 1x_1 + 2x_2 = 0 \end{cases}$$

SOLUTION



THE COMMON METHOD	THE COMMON METHOD EXPRESSED WITH MATRICES	GAUSSIAN ELIMINATION
$\begin{cases} 3x_1 + 1x_2 = 1 \\ 1x_1 + 2x_2 = 0 \end{cases}$ Start by multiplying the top equation by 2.	$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$
$\begin{cases} 6x_1 + 2x_2 = 2 \\ 1x_1 + 2x_2 = 0 \end{cases}$ Subtract the bottom equation from the top equation.	$\begin{pmatrix} 6 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 6 & 2 & 2 \\ 1 & 2 & 0 \end{pmatrix}$
$\begin{cases} 5x_1 + 0x_2 = 2 \\ 1x_1 + 2x_2 = 0 \end{cases}$ Multiply the bottom equation by 5.	$\begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$
$\begin{cases} 5x_1 + 0x_2 = 2 \\ 5x_1 + 10x_2 = 0 \end{cases}$ Subtract the top equation from the bottom equation.	$\begin{pmatrix} 5 & 0 \\ 5 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & 2 \\ 5 & 10 & 0 \end{pmatrix}$
$\begin{cases} 5x_1 + 0x_2 = 2 \\ 0x_1 + 10x_2 = -2 \end{cases}$ Divide the top equation by 5 and the bottom by 10.	$\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & 2 \\ 0 & 10 & -2 \end{pmatrix}$
$\begin{cases} 1x_1 + 0x_2 = \frac{2}{5} \\ 0x_1 + 1x_2 = -\frac{1}{5} \end{cases}$ And we're done!	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & \frac{2}{5} \\ 0 & 1 & -\frac{1}{5} \end{pmatrix}$

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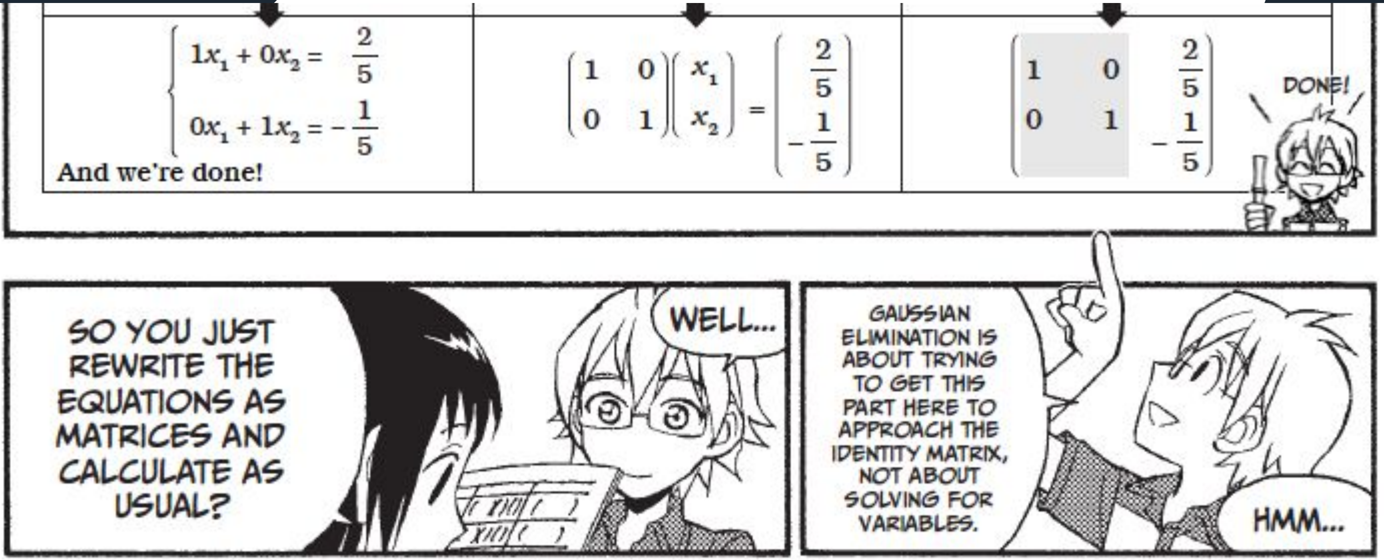

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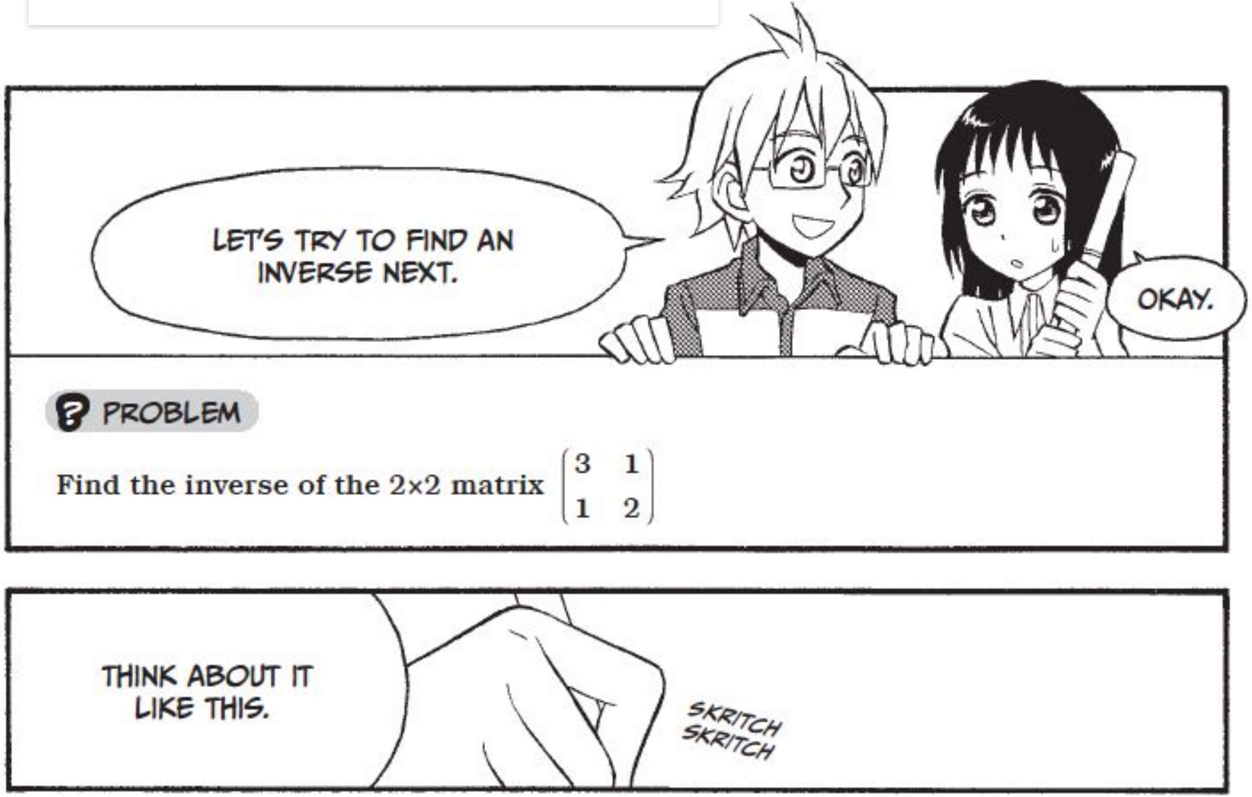
Matrix Inversion



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We're trying to find the inverse of $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$

↓

We need to find the matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ that satisfies $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

↓

or $\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ and $\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$ that satisfy $\begin{cases} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$

↓

We need to solve the systems $\begin{cases} 3x_{11} + 1x_{21} = 1 \\ 1x_{11} + 2x_{21} = 0 \end{cases}$ and $\begin{cases} 3x_{12} + 1x_{22} = 0 \\ 1x_{12} + 2x_{22} = 1 \end{cases}$

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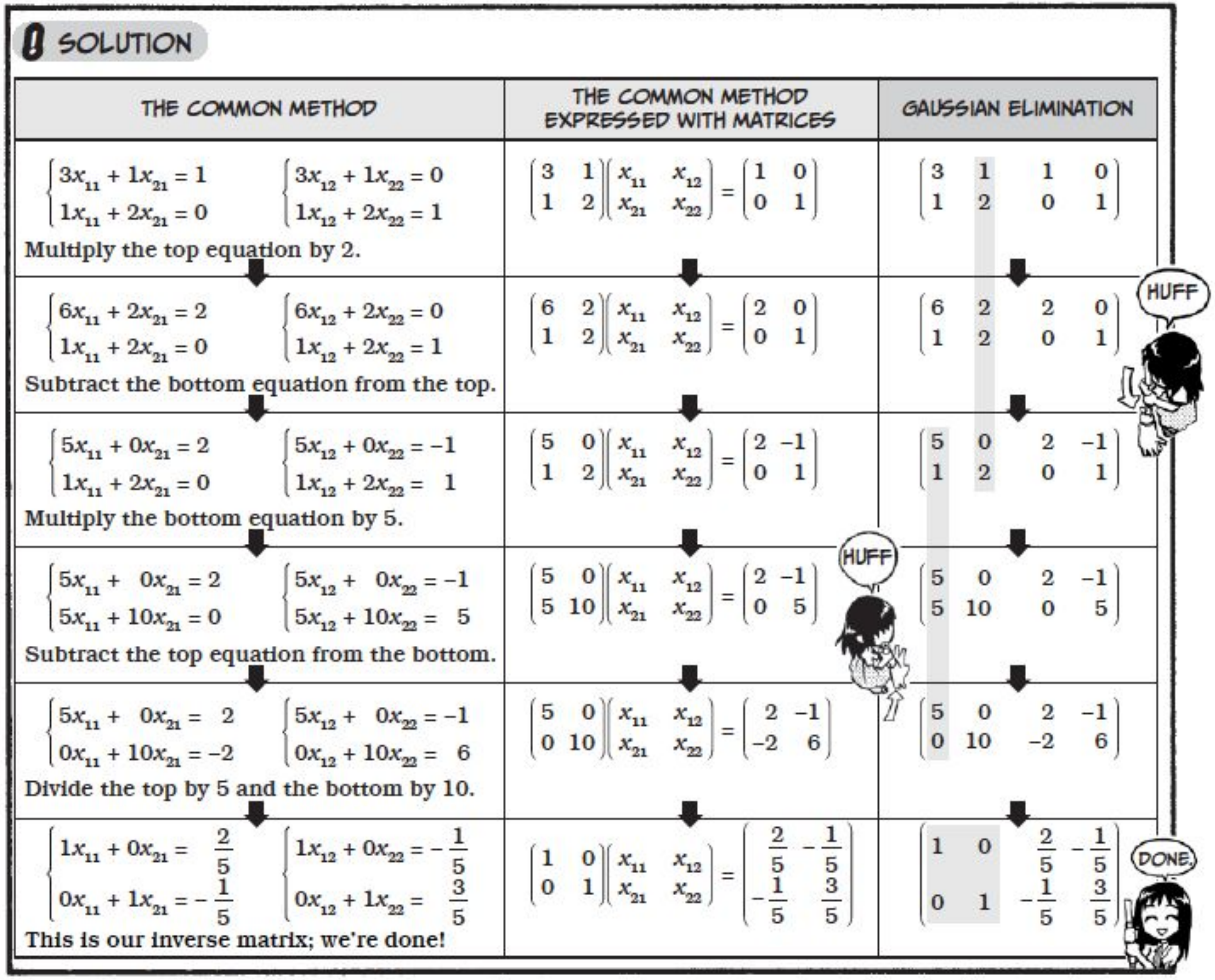

SOLUTION

THE COMMON METHOD	THE COMMON METHOD EXPRESSED WITH MATRICES	GAUSSIAN ELIMINATION
$\begin{cases}3x_{11}+1x_{21}=1\\1x_{11}+2x_{21}=0\end{cases}$ $\begin{cases}3x_{12}+1x_{22}=0\\1x_{12}+2x_{22}=1\end{cases}$ Multiply the top equation by 2.	$\begin{pmatrix}3&1\\1&2\end{pmatrix}\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}3&1&1&0\\1&2&0&1\end{pmatrix}$
$\begin{cases}6x_{11}+2x_{21}=2\\1x_{11}+2x_{21}=0\end{cases}$ $\begin{cases}6x_{12}+2x_{22}=0\\1x_{12}+2x_{22}=1\end{cases}$ Subtract the bottom equation from the top.	$\begin{pmatrix}6&2\\1&2\end{pmatrix}\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}=\begin{pmatrix}2&0\\0&1\end{pmatrix}$	$\begin{pmatrix}6&2&2&0\\1&2&0&1\end{pmatrix}$
$\begin{cases}5x_{11}+0x_{21}=2\\1x_{11}+2x_{21}=0\end{cases}$ $\begin{cases}5x_{12}+0x_{22}=-1\\1x_{12}+2x_{22}=1\end{cases}$ Multiply the bottom equation by 5.	$\begin{pmatrix}5&0\\1&2\end{pmatrix}\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}=\begin{pmatrix}2&-1\\0&1\end{pmatrix}$	$\begin{pmatrix}5&0&2&-1\\1&2&0&1\end{pmatrix}$
$\begin{cases}5x_{11}+0x_{21}=2\\5x_{11}+10x_{21}=0\end{cases}$ $\begin{cases}5x_{12}+0x_{22}=-1\\5x_{12}+10x_{22}=5\end{cases}$ Subtract the top equation from the bottom.	$\begin{pmatrix}5&0\\5&10\end{pmatrix}\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}=\begin{pmatrix}2&-1\\0&5\end{pmatrix}$	$\begin{pmatrix}5&0&2&-1\\5&10&0&5\end{pmatrix}$
$\begin{cases}5x_{11}+0x_{21}=2\\0x_{11}+10x_{21}=-2\end{cases}$ $\begin{cases}5x_{12}+0x_{22}=-1\\0x_{12}+10x_{22}=6\end{cases}$ Divide the top by 5 and the bottom by 10.	$\begin{pmatrix}5&0\\0&10\end{pmatrix}\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}=\begin{pmatrix}2&-1\\-2&6\end{pmatrix}$	$\begin{pmatrix}5&0&2&-1\\0&10&-2&6\end{pmatrix}$
$\begin{cases}1x_{11}+0x_{21}=\frac{2}{5}\\0x_{11}+1x_{21}=-\frac{1}{5}\end{cases}$ $\begin{cases}1x_{12}+0x_{22}=-\frac{1}{5}\\0x_{12}+1x_{22}=\frac{3}{5}\end{cases}$ This is our inverse matrix; we're done!	$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}=\begin{pmatrix}\frac{2}{5}&-\frac{1}{5}\\-\frac{1}{5}&\frac{3}{5}\end{pmatrix}$	$\begin{pmatrix}1&0&\frac{2}{5}&-\frac{1}{5}\\0&1&-\frac{1}{5}&\frac{3}{5}\end{pmatrix}$

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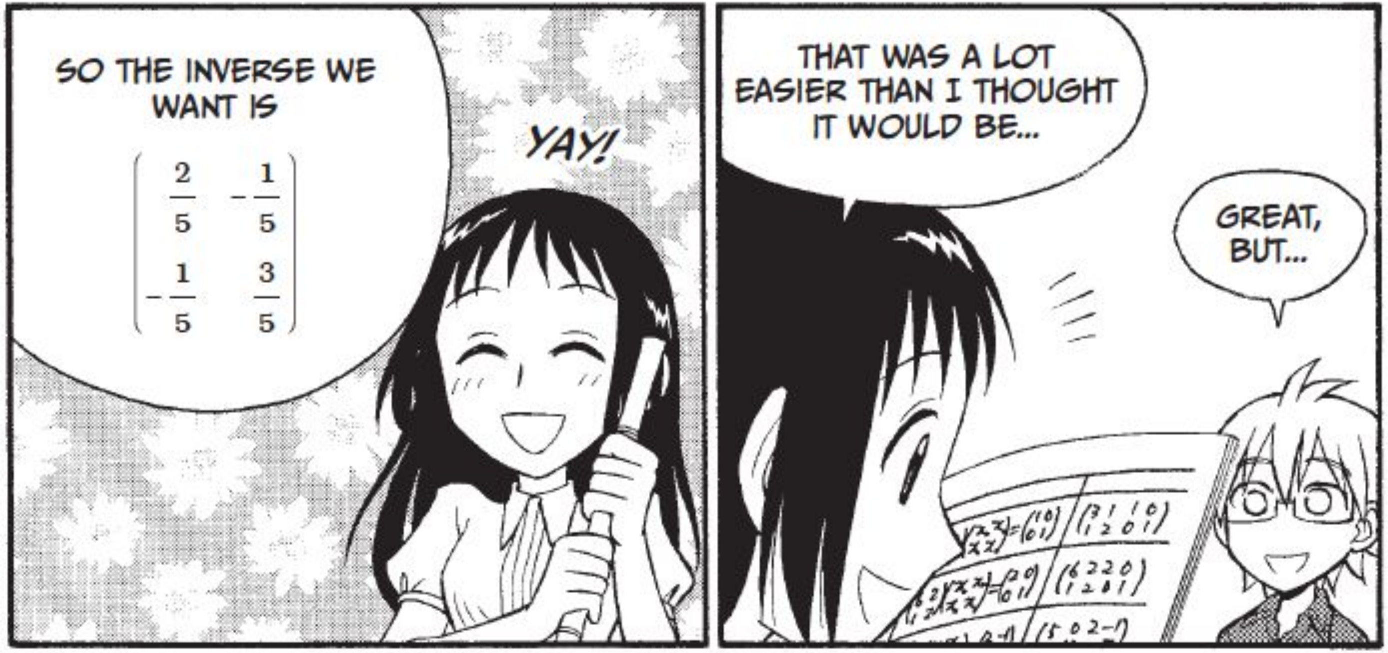

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The product of the original and inverse matrix is

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 3 \cdot \frac{2}{5} + 1 \cdot \left(-\frac{1}{5}\right) & 3 \cdot \left(-\frac{1}{5}\right) + 1 \cdot \frac{3}{5} \\ 1 \cdot \frac{2}{5} + 2 \cdot \left(-\frac{1}{5}\right) & 1 \cdot \left(-\frac{1}{5}\right) + 2 \cdot \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The product of the inverse and original matrix is

$$\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \cdot 3 + \left(-\frac{1}{5}\right) \cdot 1 & \frac{2}{5} \cdot 1 + \left(-\frac{1}{5}\right) \cdot 2 \\ \left(-\frac{1}{5}\right) \cdot 3 + \frac{3}{5} \cdot 1 & \left(-\frac{1}{5}\right) \cdot 1 + \frac{3}{5} \cdot 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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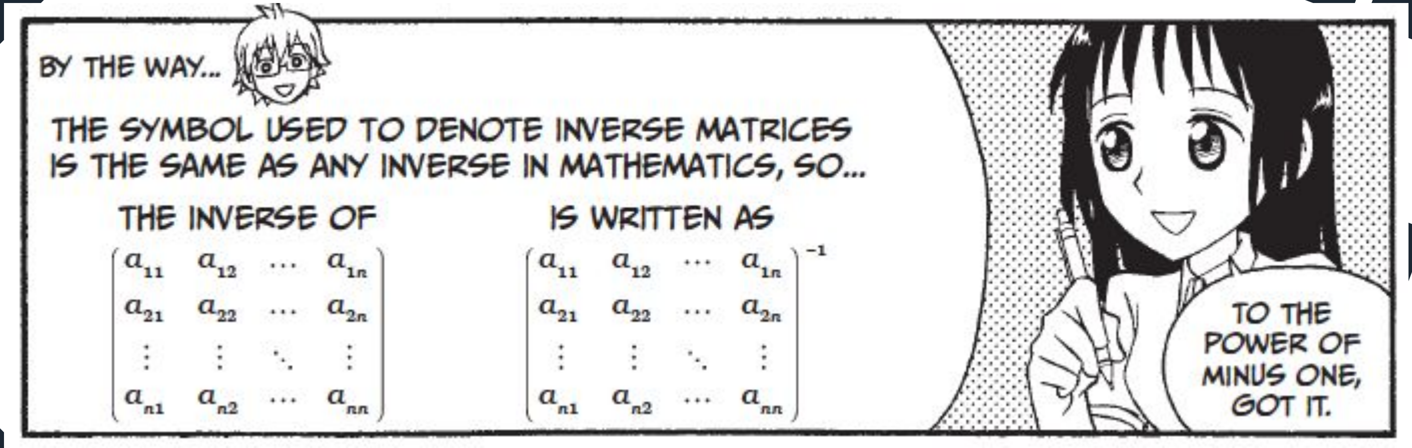

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Matrix Decomposition

Various types of decomposition.
Some examples:

- LU decomposition
- QR decomposition

LU Decomposition Example

https://youtu.be/UqemQTirijg

Please review these operations

- We will revisit these operations in the succeeding weeks, so be sure you are familiar with them again